UNIT-VIII Fourier Transforms

Dirichlet's Conditions

A function $f(x)$ is said to satisfy Dirichlet's conditions in the interval (a,b), if

(i) $f(x)$ defined and is single valued function except possibly at a finite number of points in the interval (a,b) and

(ii) $f(x)$ and $f'(x)$ are piecewise continuous in (a,b).

Fourier Integral Theorem:—

It states that $f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} \cos p(t-x) f(t)\,dt\,dp$.

Proof:— Let $f(x)$ be a function satisfying the Dirichlet's conditio[ns] in every interval $(-l, l)$ and defined as $f(x) = \frac{1}{2}[f(x+0)+f(x-$ at every point of discontinuity. We have the Fourier series of $f(x)$ given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right] \quad \text{—— (1)}$$

Where $a_0 = \frac{1}{l}\int_{-l}^{l} f(t)\,dt$

$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}\,dt$ and $b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}\,dt$.

Substituting these values of a_0, a_n, b_n in (1), we have

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l}\cos\left(\frac{n\pi t}{l}\right)\cos\left(\frac{n\pi x}{l}\right) f(t)\,dt +$$

$$\frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l}\sin\left(\frac{n\pi t}{l}\right)\sin\left(\frac{n\pi x}{l}\right) f(t)\,dt$$

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} \cos\frac{n\pi(t-x)}{l} f(t)\,dt. \quad \text{——} \ (2)$$

Let $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges i.e., it has a finite value.

Now assuming that the limit as $l \to \infty$ in (2),

We have the first term

$$\underset{l\to\infty}{Lt}\ \frac{1}{2l}\left[\int_{-l}^{l} f(t)\,dt\right] = 0.$$

$$\because \left|\frac{1}{2l}\int_{-l}^{l} f(t)\,dt\right| \le \frac{1}{2l}\int_{-l}^{l} |f(t)|\,dt.$$

Now put $\frac{\pi}{l} = \delta p$ in the second term of (2), then it takes the form

$$\frac{\delta p}{\pi}\sum_{n=1}^{\infty}\int_{-l}^{l} \cos\{n\delta p\,(t-x)\}\, f(t)\,dt.$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\int_{-l}^{l} \cos\{n\delta p(t-x)\} f(t)\,dt\right]\delta p. \quad \text{——} \ (3)$$

Now as $l \to \infty$ or $\delta p \to 0$ (3) becomes

$$\frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty} \cos\{p(t-x)\} f(t)\,dt\right]\delta p.$$

since $\underset{\delta p\to 0}{Lt}\ \sum_{n=1}^{\infty}\phi(n\delta p)\,\delta p = \int_{0}^{\infty}\phi(p)\,dp.$

Thus as $l \to \infty$ (2) takes the form

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} \cos p(t-x)\, f(t)\,dt\,dp. \quad \text{——} \ (4)$$

$\therefore$ Eqn (4) of $f(x)$ is known as Fourier integral of $f(x)$.

Fourier sine and cosine integral

we have $f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty \cos p(t-x) f(t)\, dt\, dp$.

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty [\cos(pt)\cos(px) + \sin(pt)\sin(px)] f(t)\, dt\, dp.$$

$$= \frac{1}{\pi} \int_0^\infty \cos px \int_{-\infty}^\infty \cos pt \cdot f(t)\, dt\, dp + \frac{1}{\pi} \int_0^\infty \sin px \int_{-\infty}^\infty \sin pt \cdot f(t)\, dt\, dp$$

when $f(t)$ is an odd function

$\cos pt \cdot f(t)$ is an odd function. and $\sin pt \cdot f(t)$ is even fun.

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty \sin px \int_0^\infty f(t) \sin pt \cdot dt\, dp.$$

Which is known as "Fourier sine integral".

In same way $f(t)$ is an even function

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos px \int_0^\infty f(t) \cos pt \cdot dt\, dp.$$

Which is known as "Fourier cosine integral".

* Using Fourier integral, S.T. $\int_0^\infty \frac{1-\cos \pi\lambda}{\lambda} \sin x\lambda \, d\lambda = \begin{cases} \pi/2, & \text{if } 0<x<\pi \\ 0, & \text{if } x>\pi \end{cases}$

Sol:- Let $f(x) = \begin{cases} \pi/2, & \text{if } 0<x<\pi \\ 0, & \text{if } x>\pi \end{cases}$

$$f(t) = \begin{cases} \pi/2, & \text{if } 0<t<\pi \\ 0, & \text{if } t>\pi \end{cases}$$

We K.T. the Fourier series of $f(x)$ given by

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin px \int_0^\infty f(t) \sin pt\, dt \cdot dp$$

Put $p = \lambda$.

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t)\sin\lambda t\, dt \cdot d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \left[\int_0^\pi f(t)\sin\lambda t\, dt + \int_\pi^\infty f(t)\cdot\sin\lambda t\, dt\right] d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \left(\int_0^\pi \pi/_2 \sin\lambda t\, dt + 0\right] d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \cdot \pi/_2 \left[-\frac{\cos\lambda t}{\lambda}\right]_0^\pi d\lambda$$

$$= \frac{1}{\lambda}\int_0^\infty \sin\lambda x \cdot \left[-\cos\lambda\pi + 1\right] d\lambda$$

$$f(x). = \int_0^\infty \left(\frac{1-\cos\lambda\pi}{\lambda}\right) \cdot \sin\lambda x \cdot d\lambda.$$

$$\therefore \int_0^\infty \left(\frac{1-\cos\lambda\pi}{\lambda}\right)\sin\lambda x\, d\lambda = \begin{cases} \frac{\pi}{2}, & \text{if } 0<x<\pi \\ 0, & \text{if } x>\pi \end{cases}$$

Finite And Infinite Fourier Transforms and Inverse Transforms

The complex form of fourier integral of any function $f(x)$ is in the form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty e^{ip(t-x)} f(t)\, dt\, dp.$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty e^{-ipx}\, dp \int_{-\infty}^\infty e^{ipt} \cdot f(t)\, dt$$

Now write $F(p) = \int_{-\infty}^\infty e^{ipt} f(t)\, dt$, Then $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(p) \cdot e^{-ipx} \, dp$.

Here the function $F(p)$ is called the Fourier transform of $f(x)$, and the function $f(x)$ is called the inverse Fourier transform of $F(p)$

The Infinite Fourier Transform of $f(x)$

Def:- 1. The fourier Transform of a function $f(x)$ is given by

$$F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} f(x) \, e^{ipx} \, dx.$$

The inverse Fourier transform of $F(p)$ is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(p) \cdot e^{-ipx} \, dp.$$

Fourier sine Transform

We have $f(x) = \frac{2}{\pi} \int_0^{\infty} \sin px \int_0^{\infty} \sin pt \cdot f(t) \, dt \, dp$:

Fourier sine integral

Write $F_s(p) = \int_0^{\infty} f(x) \sin px \, dx$.

Then $f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(p) \cdot \sin(px) \, dp$.

Then function "$F_s(p)$" is defined to be the infinite "Fourier sine transform" of $f(x)$. Where $0 < x < \infty$.

The function "$f(x)$" is called the inverse "fourier sine transform" of $F_s(p)$.

→ The finite Fourier sine Transform of $f(x)$ when $0<x<l$ is defined as

$$F_S\{f(x)\} = F_S(n) = \int_0^l f(x)\cdot \sin\left(\frac{n\pi x}{l}\right) dx.$$

where n is an integer and the function $f(x)$ is given by

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_S(n) \sin\left(\frac{n\pi x}{l}\right).$$

is called Inverse Finite Fourier sine transform of $F_S(n)$.

Fourier cosine Transform

We have Fourier cosine integral

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \cos(px) \int_0^{\infty} \cos(pt) f(t)\, dt\, dp.$$

Write $F_C(p) = \int_0^{\infty} \cos(px) f(x)\, dx.$

Then $f(x) = \frac{2}{\pi}\int_0^{\infty} F_C(p)\cos(px)\, dp.$

The Function $F_C(p)$ is defined to be "Infinite Fourier Cosine transform" of $f(x)$, where $0<x<\infty$

The Function $f(x)$ is called the "Inverse cosine transform" of $F_C(p)$.

→ The Finite Fourier cosine transform of $f(x)$ where $0<x<l$ is defined as $F_C(n) = \int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right) dx$ where $n\in\mathbb{Z}$

and $f(x) = \frac{1}{l}F_C(0) + \frac{2}{l}\sum_{n=1}^{\infty} F_C(n)\cos\left(\frac{n\pi x}{l}\right)$ is called Inverse Finite Fourier cosine transform of $F_C(n)$.

Properties

1. **Linearity property** :– If $F(p)$ and $G(p)$ are Fourier Transforms $f(x)$ and $g(x)$ respectively. Then

$$F[af(x)+bg(x)] = aF(p)+bG(p).$$

$[\because F[f(x)] = F(p)]$

Note:– $F_s\{af(x)+bg(x)\} = aF_s(p)+bG_s(p).$

$F_c\{af(x)+bg(x)\} = aF_c(p)+bG_c(p).$

2. **Change of scale property**:–

If $F(p)$ is the complex Fourier transform of $f(x)$ then Fourier transform of $f(ax) = \frac{1}{a}F[p/a]$.

i.e., $F\{f(ax)\} = \frac{1}{a}F(p/a)$, $a>0$.

3. **Shifting property**:– If $F(p)$ is the complex Fourier transform of $f(x)$, then the complex Fourier transform of $f(x-a)$ is

$$F[f(x-a)] = e^{ipa}.F(p).$$

4. **Modulation theorem**:– If $F(p)$ is the Complex Fourier transform of $f(x)$, then the complex Fourier transform of $f(x)\cos ax$ is

$$F[f(x)\cos ax] = \frac{1}{2}[F(p+a)+F(p-a)].$$

5. $F\{x^n f(x)\} = (-i)^n \frac{d^nF}{dp^n}.$

6. $F\{\frac{d^n}{dx^n}f(x)\} = (-ip)^n F(p).$

7. $F_s\{f'(x)\} = -pF_c(p).$

8. $F_c\{f'(x)\} = -f(0)+pF_s(p).$

* Find the Fourier transform of $f(x)$ defined by

$$f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x|>1 \end{cases}$$

Hence Evaluate (i) $\displaystyle\int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos \frac{x}{2}\, dx$.

(ii) $\displaystyle\int_0^{\infty} \frac{x\cos x - \sin x}{x^3}\, dx$.

Sol: We have $\displaystyle F(f(x)) = \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx.$

$$= \int_{-\infty}^{-1} e^{ipx}.\,0\,dx + \int_{-1}^{1} e^{ipx}(1-x^2)\,dx + \int_{1}^{\infty} e^{ipx}.\,0\,dx$$

$$= \int_{-1}^{1} \cancel{e^{ip}}(1-x^2)e^{ipx}\,dx$$

$$= \left[(1-x^2)\frac{e^{ipx}}{ip}\right]_{-1}^{1} + 2\int_{-1}^{1} x.\frac{e^{ipx}}{ip}\,dx$$

$$= +\frac{2}{ip}\left\{\left(x\frac{e^{ipx}}{ip}\right)_{-1}^{1} - \int_{-1}^{1} 1.\frac{e^{ipx}}{ip}\,dx\right]$$

$$= +\frac{2}{ip}\left[\frac{e^{ip}+e^{-ip}}{ip} - \frac{1}{ip}\left(\frac{e^{ipx}}{ip}\right)_{-1}^{1}\right]$$

$$= \frac{2}{ip}\left[\frac{2\cos p}{ip} - \frac{1}{(ip)^2}\left[e^{ip}-e^{-ip}\right]\right]$$

$$= \frac{2}{ip}\left[\frac{2\cos p}{ip} + \frac{1}{p^2}\,2i\sin p\right]$$

$$= 2\left[\frac{2\cos p}{-p^2} + \frac{2\sin p}{p^3}\right]$$

$$= 4\left[\frac{-p\cos p + \sin p}{p^3}\right]$$

$$= \frac{-4}{p^3}(p\cos p - \sin p)$$

$$\therefore \; F\{f(x)\} = \frac{4}{p^3}(\sin p - p\cos p).$$

(i) Now by inversion formula, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)\cdot e^{-ipx}\, dp.$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)\, e^{-ipx}\, dp = \begin{cases} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{cases}$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-4}{p^3}(p\cos p - \sin p)\cdot e^{-ipx}\, dp = \begin{cases} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{cases} \quad \text{— (1)}$$

Putting $x = 1/2$, we get

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{p^3}(p\cos p - \sin p)\, e^{-ip/2}\, dp = 1-\frac{1}{4} = 3/4$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{4}{p^3}(p\cos p - \sin p)\,(\cos p/2 - i\sin p/2)\, dp = -\frac{3}{8}\pi$$

Equating real parts

$$\int_{-\infty}^{\infty} \frac{p\cos p - \sin p}{p^3}\cdot \cos p/2\, dp = -3\pi/8$$

$$\Rightarrow \; 2\int_{0}^{\infty} \frac{p\cos p - \sin p}{p^3}\cos p/2 \cdot dp = -\frac{3\pi}{8}$$

$$\Rightarrow \; \int_{0}^{\infty} \frac{x\cos x - \sin x}{x^3}\cdot \cos x/2\, dx = -3\pi/8.$$

(ii) Putting $x=0$ in ①

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-4}{p^3}(p\cos p - \sin p)\,dp = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{p\cos p - \sin p}{p^3}\,dp = -\pi/2$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{p\cos p - \sin p}{p^3}\,dp = -\pi/2 \qquad (\because \text{Integrand is even})$$

$$\Rightarrow \int_{0}^{\infty} \frac{x\cos x - \sin x}{x^3}\,dx = -\frac{\pi}{4}.$$

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* Find Fourier cosine and sine transform of e^{-ax}, $a>0$. and hence deduce the inversion formula (or) Deduce the integrals (i) $\int_0^{\infty} \frac{\cos px}{a^2+p^2}\,dp$ (ii) $\int_0^{\infty} \frac{p\sin px}{a^2+p^2}\,dp$.

Sol:- Let $f(x) = e^{-ax}$, Then

The Fourier cosine transform of $f(x)$ is

$$F_c\{f(x)\} = \int_0^{\infty} f(x)\cos px\,dx.$$

$$= \int_0^{\infty} e^{-ax}.\cos px\,dx$$

$$= \left\{\frac{e^{-ax}}{a^2+p^2}\left[-a\cos px + p\sin px\right]\right\}_0^{\infty}$$

$$= 0 - \frac{1}{a^2+p^2}(-a+0) = \frac{a}{a^2+p^2}$$

And Fourier sine transform of $f(x)$ is

$$F_s\{f(x)\} = \int_0^{\infty} f(x)\sin px\,dx.$$

$$= \int_0^\infty e^{-ax} . \sin px \, dx$$

$$= \left\{ \frac{e^{-ax}}{a^2+p^2} \left[-a\sin px - p\cos px \right] \right\}_0^\infty$$

$$= \frac{p}{a^2+p^2}$$

By Inverse Fourier cosine transform, we get

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c\{f(x)\} \cos px \, dp .$$

$$= \frac{2}{\pi} \int_0^\infty \frac{a}{a^2+p^2} \cos px \, dp .$$

$$e^{-ax} = \frac{2a}{\pi} \int_0^\infty \frac{\cos px}{a^2+p^2} dp .$$

$$\Rightarrow \int_0^\infty \frac{\cos px}{a^2+p^2} dx = \frac{\pi}{2a} . e^{-ax} .$$

Now by the inverse Fourier sine transform

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s\{f(x)\} . \sin px \, dp .$$

$$e^{-ax} . = \frac{2}{\pi} \int_0^\infty \frac{p}{a^2+p^2} \sin px \, dp$$

$$\therefore \int_0^\infty \frac{p . \sin px}{a^2+p^2} dp = \frac{\pi}{2} e^{-ax} .$$

* Find the Fourier sine and Cosine transform of $2e^{-5x}+5e^{-2x}$.

Sol: Let $f(x) = 2e^{-5x}+5e^{-2x}$

The Fourier sine transform of $f(x)$ is given

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin px\,dx$$

$$= \int_0^\infty (2e^{-5x}+5e^{-2x})\sin px\,dx$$

$$= 2\int_0^\infty e^{-5x}\sin px\,dx + 5\int_0^\infty e^{-2x}\sin px\,dx.$$

$$= 2\left[\frac{e^{-5x}}{p^2+25}\{-5\sin px - p\cos px\}\right]_0^\infty + 5\left[\frac{e^{-2x}}{p^2+4}\{-2\sin px - p\cos px\}\right]_0^\infty$$

$$= 2\left[0-\frac{1}{p^2+25}(-p)\right] + 5\left[0-\frac{1}{p^2+4}(-p)\right]$$

$$= \frac{2p}{p^2+25} + \frac{5p}{p^2+4}$$

H.W.

Fourier Cosine transform

* S.T. the Fourier sine transform of

$$f(x) = \begin{cases} x & \text{for } 0<x<1 \\ 2-x & \text{for } 1<x<2 \\ 0 & \text{for } x>2 \end{cases}$$ is $2\sin s(1-\cos s)/s^2$.

Sol:- $$F_s\{f(x)\} = \int_0^\infty f(x)\sin px\,dx$$

$$= \int_0^1 f(x)\sin px\,dx + \int_1^2 f(x)\sin px\,dx + \int_2^\infty f(x)\sin px\,dx.$$

$$= \int_0^1 x \sin px\,dx + \int_1^2 (2-x)\sin px\,dx.$$

$$= \left[\frac{-x\cos px}{p}\right]_0^1 - \int_0^1 \frac{-\cos px}{p}\,dx + \left[-(2-x)\cdot\frac{\cos px}{p}\right]_1^2 - \int_1^2 (-1)\left(\frac{-\cos px}{p}\right)$$

$$= \frac{-\cos p}{p} + \frac{1}{p}\left[\frac{\sin px}{p}\right]_0^1 + \left[0 + \frac{\cos p}{p}\right] - \frac{1}{p}\left(\frac{\sin px}{p}\right]_1^2$$

$$= \cancel{\frac{-\cos p}{p}} + \frac{1}{p}\left[\frac{\sin p}{p}\right] + \cancel{\frac{\cos p}{p}} - \frac{1}{p^2}(\sin 2p - \sin p)$$

$$= \frac{1}{p^2}[\sin p - \sin 2p + \sin p]$$

$$= \frac{1}{p^2}[2\sin p - 2\sin p\cos p]$$

$$= \frac{2\sin p\,(1-\cos p)}{p^2}$$

Put $p = s$.

$$F_s\{f(x)\} = \frac{2\sin s\,(1-\cos s)}{s^2}$$

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* Find the Fourier sine transform of $\frac{x}{a^2+x^2}$ and Fourier cosine transform of $\frac{1}{a^2+x^2}$.

Sol: we have $F_s\{e^{-ax}\} = \frac{p}{a^2+p^2}$

Inverse Fourier sine transform of e^{-ax} is

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty F_s\{e^{-ax}\}\cdot \sin px\,dp.$$

$$= \frac{2}{\pi}\int_0^\infty \frac{p}{a^2+p^2}\sin px\,dp.$$

$$\int_0^\infty \frac{p}{a^2+p^2} \cdot \sin px \, dp = \frac{\pi}{2} e^{-ax}.$$

changing 'p' to 'x' & 'x' to 'p'.

$$\int_0^\infty \frac{x}{a^2+x^2} \sin px \, dx = \frac{\pi}{2} e^{-ap}.$$

$$\therefore \int_0^\infty F_s\{\cancel{f(x)}\} \cancel{\sin px \, dx} = \cancel{\frac{\pi}{2} e^{-ap}}.$$

Hence $F_s\left\{ \frac{x}{a^2+x^2} \right\} = \frac{\pi}{2} e^{-ap}$.

Illy for cosine.

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* Find the Fourier cosine transform of

(a) $e^{-ax} \cos ax$ (b) $e^{-ax} \sin ax$.

Sol: Let $f(x) = e^{-ax} \cos ax$.

$$F_c\{f(x)\} = \int_0^\infty f(x) \cos px \, dx$$

$$= \int_0^\infty e^{-ax} \cos ax \cdot \cos px \, dx.$$

$$= \frac{1}{2} \int_0^\infty e^{-ax} \cdot [\cos(a+p)x + \cos(a-p)x] \, dx.$$

$$= \frac{1}{2} \int_0^\infty [e^{-ax} \cos(a+p)x + e^{-ax} \cos(a-p)x] \, dx$$

$$= \cancel{\frac{1}{2}} \left[\frac{\cancel{e^{-ax}}}{a^2 \cancel{+} (a+p)^2} \left(\frac{-\cancel{a\cos(a+p)x}}{a+\cancel{p}} + (a\cancel{+p}) \right) \right.$$

$$\left(\because \int_0^\infty e^{-ax} \cos bx = \frac{a}{a^2+b^2} \right)$$

$$= \frac{1}{2} \left[\frac{a}{a^2+(a+p)^2} + \frac{a}{a^2+(p-a)^2} \right]$$

$$= \frac{a}{2}\left[\frac{a^2+(p-a)^2+a^2+(p+a)^2}{[a^2+(p+a)^2][a^2+(p-a)^2]}\right]$$

$$= \frac{a}{2}\left[\frac{2a^2+2p^2+2a^2}{(p^2+2ap+2a^2)(p^2-2ap+2a^2)}\right]$$

$$= \frac{a(p^2+2a^2)}{(p^2+2ap+2a^2)(p^2-2ap+2a^2)}.$$

* Find the finite Fourier sine and cosine transform of $f(x) = \sin ax$ in $(0,\pi)$.

Sol: <u>Finite Fourier sine transform</u>

$$F_s\{f(x)\} = F_s(n) = \int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

$$= \int_0^\pi \sin ax.\sin\left(\frac{n\pi x}{\pi}\right)dx = \int_0^\pi \sin ax.\sin nx\,dx \quad \text{— (1)}$$

$$= \frac{1}{2}\int_0^\pi 2\sin ax\,\sin nx\,dx$$

$$= \frac{1}{2}\int_0^\pi [\cos(a-n)x - \cos(a+n)x]\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(a-n)x}{a-n} + \frac{\sin(a+n)x}{a+n}\right]_0^\pi$$

$$= \frac{1}{2}[0-0] = 0. \quad \text{if } a\neq n$$

if <u>a=n</u> $F_s\{f(x)\} = F_s\{\sin ax\} = \int_0^\pi \sin^2 nx\,dx$ [put a=n in(

$$= \frac{1}{2}\int_0^\pi 2\sin^2 nx = \frac{1}{2}\int_0^\pi (1-\cos 2nx)\,dx.$$

$$= \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_0^{\pi}$$

$$= \frac{1}{2}\left[(\pi - 0) - (0-0)\right] = \pi/2$$

$$\therefore F_s\{\sin ax\} = \begin{cases} 0, & \text{if } a \neq n, \quad a, n \text{ are integers.} \\ \pi/2, & \text{if } a = n. \end{cases}$$

Finite Fourier cosine transform

$$F_c\{f(x)\} = F_c(n) = \int_0^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \int_0^{\pi} \sin ax \cdot \cos\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right) dx.$$

$$= \frac{1}{2}\int_0^{\pi} 2\sin ax \cos nx \, dx.$$

$$= \frac{1}{2}\int_0^{\pi} \left[\sin(a+n)x + \sin(a-n)x\right] dx$$

$$= \frac{1}{2}\left[\frac{-\cos(a+n)x}{a+n} - \frac{\cos(a-n)x}{(a-n)}\right]_0^{\pi}$$

$$= -\frac{1}{2}\left[\frac{(-1)^{a+n}}{a+n} + \frac{(-1)^{a-n}}{a-n} - \frac{1}{a+n} - \frac{1}{a-n}\right]$$

$$= \begin{cases} -\frac{1}{2}(0) = 0, & \text{if } a+n,\ a-n \text{ are even.} \\ -\frac{1}{2}\left[\frac{-2}{a+n} - \frac{2}{a-n}\right], & \text{if } a+n,\ a-n \text{ are odd.} \\ = \frac{2a}{a^2 - n^2} & \end{cases}$$

* Find $f(x)$ if its finite fourier cosine transform is

(i) $F_c(n) = \dfrac{\sin\left(\frac{n\pi}{2}\right)}{2n}$, $n=1,2,3,\ldots$ and

$\pi/4$, if $n=0$. given $0<x<2\pi$

Sol: From the inverse finite fourier cosine transform,

we've $f(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} F_c(n) \cos\left(\frac{n\pi x}{l}\right)$

Given $F_c(0) = \pi/4$, $l = 2\pi$

$$f(x) = \frac{1}{2\cancel{\pi}} \cdot \frac{\cancel{\pi}}{4} + \frac{\cancel{2}}{\cancel{2}\pi} \sum_{n=1}^{\infty} \frac{1}{2n} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\cancel{\pi}x}{2\cancel{\pi}}\right)$$

$$= \frac{1}{8} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{nx}{2}\right).$$

(ii) $F_c(n) = \dfrac{\cos(2n\pi/3)}{(2n+1)^2}$, $n=1,2,3,\ldots$ and

$= 1$, if $n=0$ given $0<x<1$.

$$f(x) = \frac{1}{1} \cdot 1 + \frac{2}{1} \sum_{n=1}^{\infty} \frac{\cos(2n\pi/3)}{(2n+1)^2} \cdot \cos\left(\frac{n\pi x}{1}\right).$$

$$= 1 + 2 \sum_{n=1}^{\infty} \frac{\cos(2n\pi/3)}{(2n+1)^2} \cos(n\pi x).$$